● ピョトル・プストラゴスキ 特定准教授

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（Shapes and arithmetic – prismatic cohomology）
専門分野： ホモトピー論（Homotopy theory）
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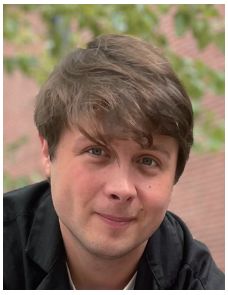

The subject of homotopy theory studies the properties of shapes which are invariant under continuous change. Despite its geometric origins, one of the great discoveries of 20th century mathematics is that such phenomena are closely related to arithmetic, the study of phenomena arising from the integers. Very informally, this means that the deep properties of prime numbers are reflected in the structure of space and the shapes which can inhibit it. The goal of this Hakubi project is the study of prismatic cohomology, which is a recently discovered invariant of varieties in mixed characteristics which led to large advances in the field of arithmetic geometry. As first shown by Hahn-Raksit-Wilson, prismatic cohomology naturally arises from even filtration, which informally means that it is essentially encoded by the structure of stable homotopy theory itself. This surprising discovery connects prismatic cohomology to a wide range of mathematics, such as the theory of motives or chromatic homotopy theory, and allows one to apply it to the study of localizing invariants of ring spectra.

What is homotopy theory?

Homotopy theory has its roots in topology, which is a classical branch of mathematics studying shapes. What makes topology slightly different from its ancient cousin geometry is that it is only concerned with those properties of shapes which remain unchanged when the shapes are stretched and deformed (but not torn or broken). It can be thought of as a kind of "soft geometry," where one studies the qualitative properties of shapes rather than quantitative ones (such as lengths or curvature). This more qualitative focus often allows one to prove surprisingly strong results using very general methods.

In homotopy theory, this focus on the "essence" of the problem is taken further, and one identifies maps (comparisons) between shapes if they can be continuously deformed into each other. For example, imagine a solid torus, the topological shape of a filled donut with a hole in the middle. From the point of view of topology, this is a different shape than the circle. However, as the torus becomes thinner and thinner, it eventually becomes a circle, so the two shapes are the same from the point of view of homotopy theory.

The use of these ideas in topology was so successful that the techniques which grew out of it (historically called "abstract homotopy theory," and now more often referred to simply as "homotopy theory") found applications in a variety of fields of mathematics, such as category theory, number theory, and algebraic geometry. Informally, modern homotopy theory is the science of identifications between objects.

Connection to number theory

As first discovered by Quillen in the 1960s, when considered from the point of view of homotopy theory, the properties of shapes can often be described in terms of an ancient branch of mathematics called number theory, or arithmetic. The latter is concerned with the study of properties of various "number systems," such as the integers.

This abstract approach to the study of homotopy theory was extraordinarily successful, and many very natural questions (such as when one shape can be built from the other using some specified operations or classification of shapes of certain type) have found beautiful answers in number-theoretic terms.

One aspect of this connection is that objects in number theory are often classified by an invariant called "height," and this classification is often reflected in homotopy theory. This allows one to break many difficult problems into more tractable ones by studying them one height at a time using a technique called chromatic filtration.

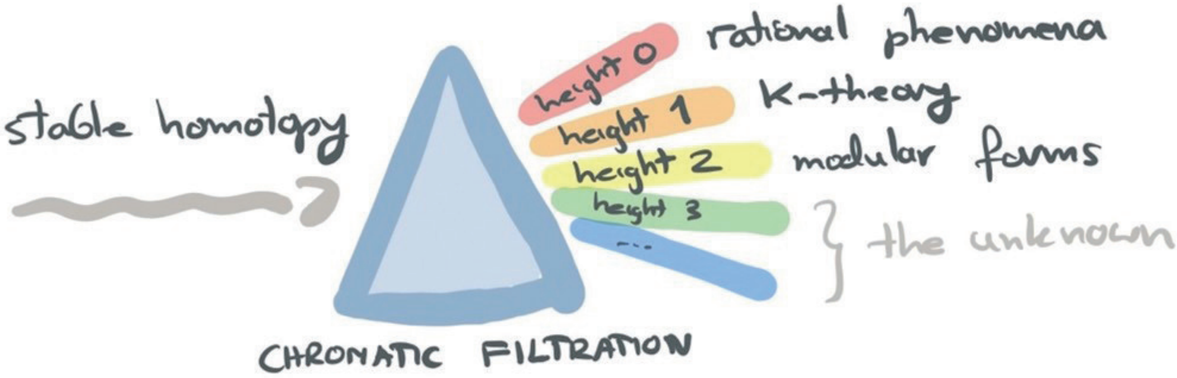

Figure 1 Chromatic filtration breaks up many phenomena in homotopy theory into parts which are easier to understand and which connect to many branches of mathematics.

Prismatic cohomology

The goal of this project is the study of prismatic cohomology. First introduced by Bhatt, Morrow, and Scholze (2019), prismatic cohomology is an invariant of number-theoretic objects known as p-adic formal schemes which was first constructed using methods of homotopy theory.

Cohomology theories are some of the most powerful methods of studying arithmetic objects, and a wide variety has been known for decades, such as etale or crystalline cohomology. Very informally, one can think of prismatic cohomology as a kind of "universal" cohomology theory which is related to the other known ones by comparison results. Because different cohomology theories detect different properties of the object in question, the study of prismatic cohomology allows one to constrain the kind of properties an object can have at the same time.

The even filtration

While the first construction of prismatic cohomology was very number-theoretic in nature, it can also be constructed from even filtration. The latter, first introduced by Hahn-Raksit-Wilson (2022), provides a canonical filtration on a ring spectrum in purely homotopy-theoretic terms.

One advantage of the construction in terms of even filtration is that it connects prismatic cohomology to a variety of other phenomena in mathematics, such as chromatic filtration or the theory of motives from algebraic geometry. Moreover, this construction is more general, allowing one to construct prismatic cohomology not only for objects of pure number theory but also for ring spectra, which are objects of mixed number-theoretic and homotopy-theoretic nature.

One large success story of this general form of prismatic cohomology was the resolution of John Rognes' "redshift conjecture," first made in 1999. However, many of the basic properties of prismatic cohomology of ring spectra and even filtration in general, such as its precise relationship to the theory of motives, remain unknown, and the main goal of this project is to answer these questions.

References:

- [1] Bhatt, B., Morrow, M. & Scholze, P. Topological Hochschild homology and integral p-adic Hodge theory. Publ.math.IHES 129, 199–310 (2019). https://doi.org/10.1007/s10240-019-00106-9
- [2] Jeremy Hahn, Arpon Raksit, and Dylan Wilson, A motivic filtration on the topological cyclic homology of commutative ring spectra, arXiv preprint arXiv:2206.11208 (2022).