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研究課題： 混標数の幾何学による既約シンプレクティック多様体の数論の研究
（Study of Aarithmetic of Iirreducible Ssymplectic Vvarieties via Mmixed Ccharacteristic Ggeometry）

専門分野： 数論幾何学（Arithmetic Geometry）

受入先部局： 理学研究科（Graduate School of Science）

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私の専門は，数論幾何学という，整数の性質を幾何学的背景から解き明かす分野です．数論幾何学の大きな問題意識に，方程式の整数解・有理数解を知るために，その方程式の定める図形の幾何学的性質を研究することが挙げられます．このためには，図形の「正標数還元」をとるというテクニックが非常に強力です．正標数還元とは，各素数 p ごとに定まる，方程式を「p で割った余りの世界」で考え直して得られる，いわば「素数による影」の様な図形であり，多くの情報を持っています．しかしながら，これらの影は元の図形とは住む世界が違い，その幾何学的な様相も大きく異なることが知られています．

私の白眉プロジェクトの研究では，既約シンプレクティック多様体という，興味深い対称性を持った図形について，近年急速に発展した混標数の幾何学というを用いることで「正標数還元」の様子を調べます．更に，これらの研究を応用することで，より一般の図形の正標数還元の考察や，整数解の有限性を始めとした整数論への応用を目指します．

My specialty is arithmetic geometry, a field that aims to uncover the properties of integers through a geometric perspective.
One of the central concerns of arithmetic geometry is studying the geometric properties of the shapes defined by equations in order to understand the integer/rational solutions of those equations.
An important ingredient in this pursuit is the technique so-called “mod p reduction”, which means rethinking equations in “the world of remainders modulo each prime number p”, yielding shapes that are, in a sense, “shadows cast by primes p" and contain a wealth of information. However, it is known that these shadows live in a completely different world from the original shapes and have significantly different geometric aspects.
In my study in the Hakubi project, I will investigate the behavior of “mod p reduction” for shapes with interesting symmetries called irreducible symplectic varieties, using revently developed theory of geometry in mixed characteristic.
I hope that this study lead us to reveal the nature of “mod p reduction” for more general varieties, and I also want to apply these ideas to number theory.

What is arithmetic geometry?

Despite being the most familiar mathematical object, integers still hold many mysterious open problems. The goal of arithmetic geometry is to solve these problems using geometric methods.
For example, let's say we want to know how many triangles with integer side lengths exist. This problem can be reduced to finding rational solutions to the equation x^2+y^2=1 by the well-known Pythagorean theorem. So, how can we determine rational solutions?
Recall that this equation defines a unit circle. The problem is then reduced to finding rational points on the unit circle.
By associating each of these points with the slope viewed from the point (-1,0), which is a rational number, we can actually obtain a one-to-one correspondence, completely describing the set of solutions.

Now, what happens when we consider other shapes besides circles,
for example, x^3+y^3=1? What happens when we add more equations? These questions have led to the development of a vast and profound theory called arithmetic geometry.

Mod p reduction of varieties

How can we extract arithmetic properties from the geometric shapes defined by equations like the one we saw earlier?
The point is that their equations have integer/rational coefficients!

This allows us to rethink equations in the world of remainders after dividing by each prime number p. The resulting new shapes are called the mod p reductions of the original shape, and they can be thought of as shadows of the original shape cast by the prime p.
Using these shadows for each prime, we can extract arithmetic properties of the shape, which is an important technique in arithmetic geometry.
Let's consider the curve defined by the equation y^2 = x^3 + 3 as an example. The list of mod p reductions is shown in the figure 1.

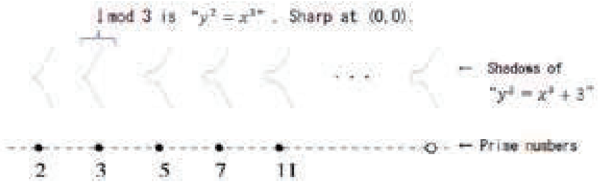

figure 1

We can see that sharp points appear when p is 2 or 3, while the curve is smooth for other primes.
Such information is very valuable. For example, it is known that there are only 784 (finiteness is important) cubic curves with the same properties (such a result is called a finiteness result). Enumerating such lists as the one shown above is one of the main goals of my research.

Irreducible symplectic varieties

Higher-dimensional shapes have much more complicated structures compared to the one-dimensional case, making it difficult to study general shapes directly. However, it is known that shapes with certain flatness can be decomposed into three types (known as the Beauville-Bogomolov decomposition): abelian varieties (generalization of cubic curves), irreducible symplectic varieties (generalization of quartic surfaces), and Calabi-Yau varieties (e.g. quintic threefolds).
Among them, the arithmetic of abelian varieties is relatively well understood and leads to a wealth of applications to number theory.
On the other hand, irreducible symplectic varieties are also shapes with very deep symmetries and are of great interest in arithmetic geometry. Figure 2 shows a two-dimensional irreducible symplectic variety, also known as a K3 surface.

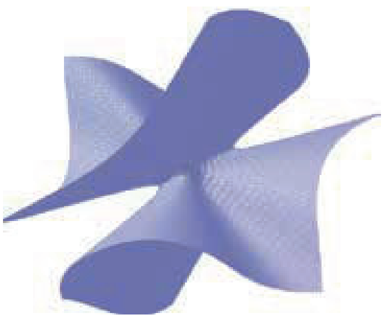

figure 2

Despite their interest, there is still not much progress in the study of irreducible symplectic varieties.\
My research aims to study the reduction of irreducible symplectic varieties and reveal their arithmetic properties.

Method:Geometry in mixed characteristic

I wanna study mod p reduction using the theory of geometry in mixed characteristic. Recall that mod p reduction lives in a completely different world, the world of residues modulo p.
Geometry in mixed charactteristic is one of the unified theories in geometry that integrates these completely different geometries, where "mixed" means that various prime numbers are mixed.
Among them, the minimal model program in mixed characteristic, which has been rapidly developing in recent years, is very powerful. The minimal model program is a grand theory originally developed for non-mixed characteristic ordinary shapes. It is the theory to find "the simplest model" of given shapes by contracting or replacing superfluous parts of them.
Theories like these are highly compatible with my goal of classifying mixed characteristic geometries and can be expected to produce significant results.
However, mixed characteristicization has been very difficult. The reason is that the concept of "singularity" which is essential in the development of geometry, had not been formulated well. However, in recent years, singularities that behave well in mixed characteristic world and mod p worlds have been defined, and mixed characteristic geometry has entered a major turning point.
In the Hakubi project, I will continue to explore the reduction of irreducible symplectic varieties while further advancing the basic theory of mixed characteristic geometry.
Moreover, I hope that this study leads us to attack more general shapes such as Calabi-Yau varieties, and I also want to apply them to number theory such as finiteness problems.

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